THE
MATHEMATICAL GAZETTE.

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OUR LOCAL BRANCH.

AT a meeting of the North Wales branch, held on May 30 at the Friars' School, Bangor, there was a satisfactory attendance. A discussion on the use of mathematical text-books in schools was opened by Mr. D. J. Williams, head-master of the Bethesda County School, who, after pointing out the advantages and disadvantages of using (1) no text-books at all, (2) books with examples only, (3) full text-books, decided in favour of a full text-book, as saving much time and giving children scope for finding things out for themselves. Several interesting suggestions were made by subsequent speakers, especially with regard to a teacher drawing up his own course of work for pupils in particular subjects. The majority were in favour of making judicious use of full text-books, which should not be too crowded with details.

Dr. Bryan gave an interesting summary of the proceedings of the Fourth International Mathematical Congress at Rome, and commented on the fact that of more than 500 present only 20 were Englishmen.

The next meeting was arranged for Nov. 21, when a discussion will take place on the teaching of algebra and logarithms.

THE GEOMETRICAL TREATMENT OF TRIGONOMETRICAL SERIES.*

IN the standard text-books on trigonometry the object of the authors is to reduce geometry to algebra, to give geometrical proofs of a few fundamental formulae and afterwards to develop the subject on algebraic lines. When alternative geometrical proofs are given they are not often designed to illustrate the steps in the algebra. De Morgan's *Double Algebra* and Hayward's *Vector Algebra* may be mentioned as works in which free use is made of geometry, but the emphasis laid on the interpretation of a vector as a complex quantity makes their method appear rather advanced. The slides which I shewed to the annual meeting were intended to illustrate the

* Notes of an address at the annual meeting, January, 1908.

advantage of using geometrical methods in obtaining the sums of trigonometrical series.

The first slide (not reproduced) explains the summation of a series of cosines

$$\sum_0^5 \cos na = \frac{\sin \frac{6a}{2}}{\sin \frac{a}{2}} \cos \frac{5a}{2}.$$

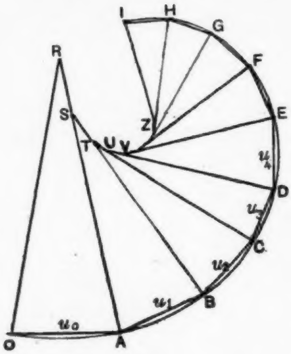


FIG. 1.

Lines of unit length are taken making angles 0, a, $2a$, ..., with an axis and placed end to end so as to form sides of a regular polygon. The angle at the centre of the polygon subtended by any one side is a, so that the radius of the circle is $\frac{1}{2}\operatorname{cosec}\frac{a}{2}$. The angle subtended by all the sides together is $\frac{6a}{2}$, so that the chord can be calculated and its projection on the axis is easily found. The diagram can also be used to illustrate the meaning of the steps in the analytical proof as the projection of each side is the difference between the projections of two radii.

The next slide (Fig. 1) shews how the convergence of a Fourier Series with positive coefficients can be investigated. The algebraical proof of this theorem given by Abel will be found in Chrystal's *Algebra*, Part II., p. 122. The enunciation is that if u_1, u_2, u_3, ..., are positive real quantities in descending order of magnitude such that $\lim_{n\to\infty} u_n = 0$, and if a is not a multiple of 2π, then $\Sigma u_n \cos na$ is a convergent series. The diagram represents the terms of the series as the projections of the sides of a polygon. The sum of the sides of the polygon may be infinite, and we require a proof that it will wind round in a spiral approaching indefinitely close to a certain focus. By bisecting the angles of the polygon a number of triangles are formed which are all similar, and as their bases are in decreasing order their sides must also decrease. The vertices of these triangles lie on a spiral which is traced always in one direction, and the length of this spiral is equal to the side of the first triangle. Since u_n tends to zero the two spirals approach each other indefinitely, and since the inner spiral is of finite length the focus of either is at a finite distance from its starting place. The steps of the proof which has just been sketched correspond to those in Abel's algebraical proof. Hayward (*loc. cit.*, p. 186) draws our first spiral and states, "in the absence of any general criterion the convergency must be investigated for each particular case."

The next slide (Fig. 2) shews a set of spirals drawn to illustrate the summation of the series

$$\sin\theta + \frac{\sin 3\theta}{3} + \frac{\sin 5\theta}{5} + \dots$$

for different values of the parameter θ. The values shewn are

$$\frac{\pi}{4},\ \frac{\pi}{8},\ \frac{\pi}{16},\ \frac{\pi}{32},\ -\frac{\pi}{32},\ -\frac{\pi}{16},\ -\frac{\pi}{8},\ -\frac{\pi}{4}.$$

This diagram illustrates very clearly the way in which a limiting value of a function of two variables depends on the order in which the two variables proceed to their limit. As long as the first side of one of the spiral polygons makes a positive finite angle with the axis, the focus is found at a distance

$\frac{\pi}{4}$ from the axis; but if the first side really lies along the axis, then the focus will be found on that strait and narrow way,

i.e. $$\lim_{\theta\to 0}\lim_{n\to\infty}\left(\sin\theta+\frac{\sin 3\theta}{3}\ldots\frac{\sin(2n-1)\theta}{2n-1}\right) \text{ is } \frac{\pi}{4},$$

whilst $$\lim_{n\to\infty}\lim_{\theta\to 0}\left(\sin\theta+\frac{\sin 3\theta}{3}\ldots\frac{\sin(2n-1)\theta}{2n-1}\right) \text{ is } 0.$$

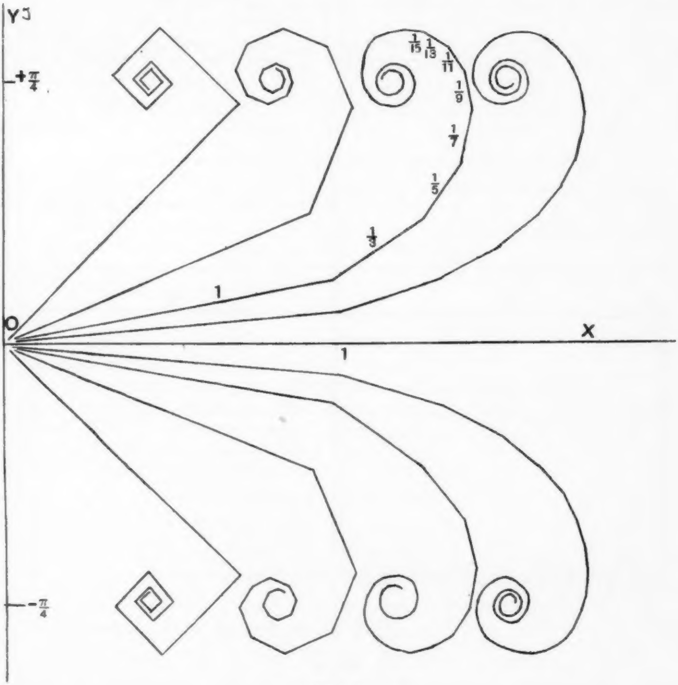


FIG. 2.

If there is a connection between θ and n, then there will be some other limit. Thus the highest points of our spirals are given by $n\theta=\frac{\pi}{2}$, so that we can write $$\lim_{\theta\to 0;\ 2n\theta=\pi}\left(\sin\theta+\ldots\frac{\sin(2n-1)\theta}{2n-1}\right)\doteqdot\cdot 94.$$

The last slide (Fig. 3) shews how the sum of eight terms of our series compares with the sum to infinity. As the number of terms is increased, the two sums shewn by the firm and dotted lines tend to coincide except near the points of discontinuity of the infinite series. The oscillations near those points retain their amplitude whilst their width diminishes when the number of terms is increased. It has been proved by Bôcher that every

discontinuity of a Fourier Series must be of this character. The case which is usually discussed in this connection is that of the series

$$\sin\theta + \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} + \dots,$$

which is not so simple as the one I have chosen, as the function is not constant between its discontinuities. The number of examples which can be

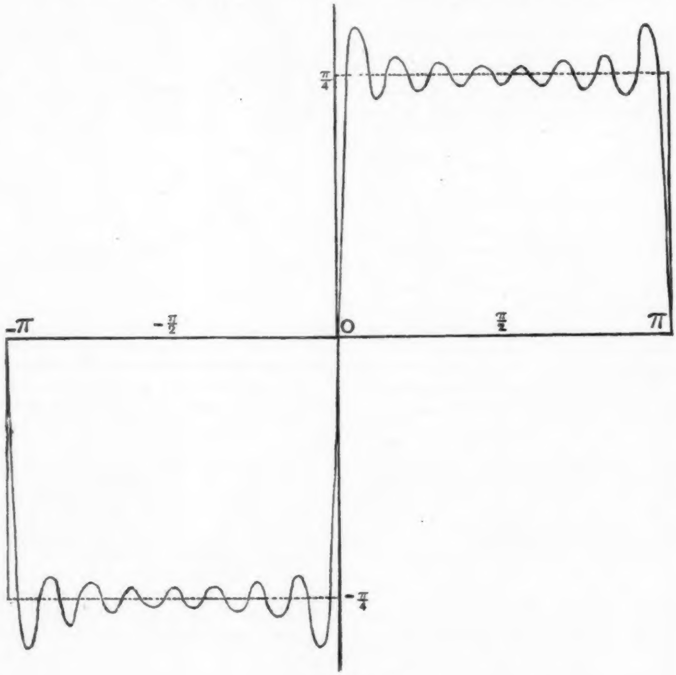


FIG. 3.

constructed is of course legion, and, although the diagrams take rather a long while to draw, I believe it will be admitted that the insight into the theory of infinite series is quite worth the trouble. F. J. W. WHIPPLE.

GENERAL THEORY OF IRRATIONAL NUMBER; INTEGRATION OF dy/y.

THERE is a perfectly natural way of arriving at the differential coefficient of a^x, and the corresponding integration of dy/y, provided that the ordinary inadequate bookwork of indices is properly supplemented to include irrational indices. (This is really necessary for logarithms to be treated rigorously.) The graph of $y=a^x$, x any real number and a positive,

is then a continuous curve for which it is easily proved that $\frac{dy}{dx}=\frac{y}{k}$, where k is a constant (the subtangent of x, y) which depends on the value of a, and may be defined as the *modulus* of the system of exponentials to *base a*. In this case e is the base whose modulus is unity, and e^a is readily found by Taylor's theorem, since

$$\frac{de^x}{dx}=e^x=\frac{d^2e^x}{dx^2}=\dots.$$

I. Something like the following chain of theorems is necessary; the base a is always considered as positive.

(i) 'If $a > 1$, a^x can be made $>$ any assigned quantity, by making x a great enough integer; and if $a < 1$, $a^x <$ any assigned quantity, etc.'

We can prove successively

$$(1+x)^2 > 1+2x\text{ ; }(1+x)^3 > (1+x)(1+2x) > 1+3x\text{ ;}$$

and so on;

$$\therefore\ (1+x)^n > 1+nx > \text{any assigned quantity, } n \text{ great enough.}$$

This proves the theorem when $a > 1$, and by using the reciprocal of a, the corresponding theorem is proved when $a < 1$.

(ii) 'If $a > 1$, n a given integer, a number $b > 1$ can be found such that $b^n < a$; and if $a < 1$, $c < 1$ can be found so that $c^n > a$.'

Without the binomial theorem we can show that the sum of coefficients of $(1+x)^n=2^n$; hence if $x<1$

$$\begin{aligned}(1+x)^n &< 1+c_1x+c_2x+c_3x+\dots c_nx\\ &< 1-x+2^nx < 1+x(2^n-1).\end{aligned}$$

By making x small enough, this can be made $< a$.

This proves when $a > 1$; by taking reciprocals, we prove when $a < 1$.

(iii) 'Every real positive number a has one only real positive n^{th} root'; this root we shall call the *principal n^{th} root* of a.

If $a > 1$, numbers b exist > 1, such that $b^n < a$.

Choose b the greatest number, such that $b^n \not> a$;

$$\therefore\ \text{either } a > b^n \text{ or } a=b^n.$$

But if $a > b^n$, and c is any number $> b$, $c^n > a$; and

$$\begin{aligned}c^n-b^n&=(c-b)(c^{n-1}+c^{n-2}b+\dots+b^{n-1}),\\ &<\text{any assigned quantity, if } c-b \text{ is small enough,}\\ &<a-b^n, \qquad \text{if } c-b \text{ is small enough.}\end{aligned}$$

But this requires $c^n < a$, which is not true;

$$\therefore\ a \not> b^n;\quad \therefore\ a=b^n.$$

Also, the value of b^n increases $(b > 1)$ as b increases, hence no other real positive quantity than b can satisfy the condition $b^n=a$.

This proves the theorem when $a > 1$, and by reciprocals we can prove when $a < 1$.

We can now define $a^{\frac{p}{q}}$ as the *principal q^{th} root* of a^p, proving the surd laws first and deducing from them the index laws for fractional indices.

The simplest method for the next stage is the decimal method of representing number, of which I have given a brief outline, sufficient for geometry, in my *Elementary Pure Geometry*.

It is easily shown that every real number a (defined as the measure of a magnitude X in terms of another Y as unit) can be *represented* by a decimal $d\cdot d_1d_2d_3\dots d_n\dots$ *ad inf.*

This decimal may be regarded *pro tem.* as a symbol or mark and not as itself a number.

M 2

If $a_n = d \cdot d_1 d_2 d_3 \ldots d_n$, terminating in the n^{th} place, and $a'_n = a_n + \frac{1}{10^n}$, obtained by adding unity to the last digit of a_n, then we may call a_n, a'_n *the n^{th} decimal approximations* of a. Also,

(iv) Every $a_n <$ any a'_n,

and $$a > a_n \text{ but } \lesseqgtr a'_n, \text{ whatever integer } n \text{ is.}$$

If the decimal $d \cdot d_1 d_2 \ldots d_n \ldots$ is given, we easily show that there is one only number $a > a_n$ but $\lesseqgtr a'_n$, so that each decimal represents one only number; and the ordinary laws of operation are easily proved.

We may now define for an irrational index x, 'a^x is intermediate between a^{x_n} and $a^{x'_n}$ for all values of n, where x_n, and x'_n are the n^{th} decimal approximations of x.'

We prove first

(v) 'There is one only number, viz. a^x between a^{x_n} and $a^{x'_n}$ for all values of n.'

If $a > 1$, and b is the *least* quantity $>$ all a^{x_n}, and c is any quantity $> b$;

then $$\frac{a^{x'_n}}{a^{x_n}} = a^{\frac{1}{10^n}} \quad \left(\because \; x'_n = x_n + \frac{1}{10^n}\right).$$

But $\left(\frac{c}{b}\right)^{10^n} > a$, if n is great enough, by (i) above;

$$\therefore \; \frac{a^{x'_n}}{a^{x_n}} = a^{\frac{1}{10^n}} < \frac{c}{b}, \; n \text{ great enough};$$

$$\therefore \; c > a^{x'_n} \text{ since } b < a^{x_n};$$

$$\therefore \text{ there is one only number } b > a^{x_n} \text{ but } \lesseqgtr a^{x'_n} \text{ for all values of } n.$$

Similarly for $a < 1$.

A similar method enables us to prove the index laws for irrational indices.

Negative indices and a^0 are defined as usual.

(vi) Every real number b can be expressed as some power of a given real number a.

Some powers of a can be found $> b$ and $< b$; and we have only to take the greatest power of a, not greater than b, or some equivalent according to the values of a and b relative to unity, to have a power $a^x = b$.

If now a is any positive number > 1, a^x is a quantity which increases continuously from zero to 1 to $+\infty$ as x increases from $-\infty$ to zero to $+\infty$, and it has one real value for each value of x. The graph of a^x is therefore a continuous curve consisting of one branch.

II. Graph of $y = a^x$. $\int \frac{dy}{y}$.

If PQ, $P'Q'$ are chords of $y = a^x$, whose projections on the axis of x are equal ($MN = M'N'$), and if PQ, $P'Q'$ meet the x-axis in T, T', we may call TM, $T'M'$ the *subchords*.

Let the coordinates of P, Q, P', Q' be x, y; v, z; x', y'; v', z'; then

$$\frac{z}{y} = \frac{a^v}{a^x} = a^{v-x} = a^{v'-x'} = \frac{a^{v'}}{a^{x'}} = \frac{z'}{y'};$$

$$\therefore \; \frac{TM}{PL} = \frac{PM}{QL} = \frac{y}{z-y} = \frac{y'}{z'-y'} = \frac{P'M'}{Q'L'} = \frac{T'M'}{P'L'};$$

$$\therefore \; TM = T'M';$$

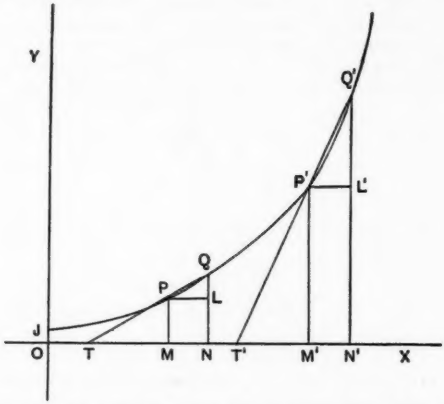
i.e. the subchord is constant when the x projection of the chord is constant, and also the ratio of ordinates is constant.

Move Q, Q' to coincidence with P, P', keeping $MN=M'N'$; then, in the limit,

$$\text{subtangent } TM=T'M'=\text{constant}=k \text{ say.}$$

Also
$$\frac{dy}{dx}=\text{lt. of } \frac{QL}{PL}=\frac{PM}{\text{subtangent}}=\frac{y}{k}=\frac{a^x}{k}.$$

It is easy to see that for different bases, this quantity k will have different values. k is the *modulus*.



We may now define e as that base whose subtangent or modulus is unity; hence, since y can be expressed as a power of e, we may write

$$y=e^x, \text{ modulus unity };$$

$$\therefore \frac{dy}{dx}=\frac{y}{1}=e^x,$$

and
$$\int\frac{dy}{y}=\int dx=x=\log_e y.$$

Again, any real quantity y can be expressed as some power, say a^x, of a, and a can be expressed as some power, say e^c, of e;

$$\therefore y=a^x=e^{cx}, \text{ and } c=\log_e a;$$

$$\therefore \frac{dy}{dx}=\frac{a^x}{k}=ce^{cx}=ca^x;$$

$$\therefore k=\frac{1}{c}=\frac{1}{\log_e a}=\log_a e.$$

We thus have the modulus of any base a, in terms of e.

Also, if $y=e^x$,

$$\frac{dy}{dx}=e^x, \quad \frac{d^2y}{dx^2}=e^x, \text{ etc.}; \quad e^0=1;$$

hence, by Taylor's theorem,

$$e^x=1+x+\frac{x^2}{1\,.\,2}+\dots,$$

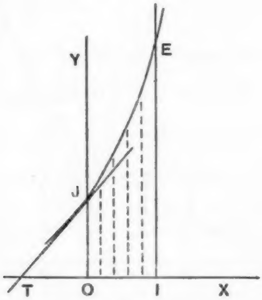
and
$$e=1+1+\frac{1}{1\,.\,2}+\dots.$$

We thus arrive quite naturally at this fundamental theorem of the Theory of Functions.

III. If n ordinates to the curve are drawn at successive intervals along the x-axis equal to $\frac{1}{n}$, the ratio of any two consecutive ordinates is constant $=1+\frac{m}{n}$ say; hence, if a is the base,

$$\left(1+\frac{m}{n}\right)^n=a \left(\text{the ordinate at dist. } n\times\frac{1}{n}\right),$$

and
$$a^x=\left(1+\frac{m}{n}\right)^{nx}.$$



If we now consider the part of the graph of $y=e^x$ from 0, 1 to 1, e, and if OI is divided into n equal parts, and the ratio of successive ordinates is $1+\frac{m}{n}$;

then the subtangent TO at $J=$ unity;

$\therefore$ the tangent of slope at J is 1, and $m>1$;

also, if n is indefinitely increased, m becomes unity in the limit ($QL=PL$ in previous figure if P, Q coincide with J);

$$\therefore\ e=\text{lt. of } \left(1+\frac{1}{n}\right)^n,\ n \text{ infinite,}$$

and
$$e^x=\text{lt. of } \left(1+\frac{1}{n}\right)^{nx},\ n \text{ infinite.}$$

We can thus connect the ordinary method of developing e with a characteristic property of the graph of $y=e^x$.

We easily derive the most satisfactory general method of differentiating x^n, when n is any real number whatever.

Let $y=x^n$, $x=e^z$; $\qquad \therefore\ y=e^{zn}$;

$$\therefore\ \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=ne^{zn}\times\frac{1}{e^z}=\frac{nx^n}{x}=nx^{n-1}.$$

The great advantage of the complete method, as I have given it, is that the continuity of a^x when x varies is fully established at the commencement; it therefore serves as an admirable example of a continuously varying function such as is implied in the fundamental processes of differentiation and integration.

E. Budden.

MATHEMATICAL NOTES.

258. [D. 2. a. α.] *A proof that Σu_n is divergent when u_n/u_{n-1} can be expanded in a series of descending integral powers of n beginning with $1-\frac{1}{n}$.*

It is well-known that when the ratio u_n/u_{n-1} can be expanded in a series of descending integral powers of n beginning with $1-\frac{a}{n}$ the convergency can be determined by comparison with $\Sigma 1/n^\kappa$ by choosing κ between a and 1. In the case in which $a=1$ the comparison with $\Sigma 1/n$ breaks down and it is apparently necessary to use the next type in the scale of convergency, viz. $\Sigma 1/n \log n$. It is, however, possible to get over the difficulty by comparing the terms of Σu_n with later terms of $\Sigma 1/n$.

Write
$$v_n=\frac{1}{n-\kappa},$$
$$v_n/v_{n-1}=1-\frac{1}{n}-\frac{\kappa}{n^2}-\frac{\kappa^2}{n^3}\dots.$$

Whatever be the value of β an integral value of κ can be found such that
$$u_n/u_{n-1}\equiv 1-\frac{1}{n}-\frac{\beta}{n^2}-\frac{\gamma}{n^3}\dots>1-\frac{1}{n}-\frac{\kappa}{n^2}-\frac{\kappa^2}{n^3}\dots$$
for all values of n above a certain finite limit.

When κ takes this value, then Σu_n diverges more rapidly than $\Sigma\frac{1}{n-\kappa}$ which is known to be a divergent series.

This theorem is not quite as general as the one enunciated by Mr. Roseveare (*Gazette*, p. 246), who omits the word 'integral' in the enunciation, but it covers all the cases of frequent occurrence.

The most important example of the theorem is the limiting case of the Hypergeometric Series for which
$$u_n=\frac{a(a+1)\dots(a+n-1)\beta(\beta+1)\dots(\beta+n-1)}{\gamma(\gamma+1)\dots(\gamma+n-1)\,\delta(\delta+1)\dots(\delta+n-1)}$$
$$u_n/u_{n-1}=\frac{(a+n-1)(\beta+n-1)}{(\gamma+n-1)(\delta+n-1)}$$
$$=\left(1+\frac{a-1}{n}\right)\left(1+\frac{\beta-1}{n}\right)\left(1+\frac{\gamma-1}{n}\right)^{-1}\left(1+\frac{\delta-1}{n}\right)^{-1}$$
$$=1+\frac{a+\beta-\gamma-\delta}{n}+\dots,$$
an expansion in negative integral powers of n. Our theorem shews us accordingly that the Hypergeometric Series is divergent when $a+\beta-\gamma-\delta=-1$. (Cf. Chrystal, Part II., p. 116, where the higher criterion is used.) As a numerical example we may take the case $a=-\frac{1}{4}$, $\beta=\frac{3}{4}$, $\gamma=\frac{1}{2}$, $\delta=1$,
$$u_n=-\frac{[3.7.11\dots(4n-5)]^2(4n-1)}{2.4.6.8\dots4n}$$
$$u_n/u_{n-1}=1-\frac{1}{n}-\frac{3}{16}\cdot\frac{1}{n^2}.$$

In this case the series diverges less rapidly than $\Sigma\frac{1}{n}$ but more rapidly than $\Sigma\frac{1}{n-1}$ and it is therefore divergent.

F. J. W. WHIPPLE.

259. [V. a.] *On Stolz and Gmeiner's Proof of the Sine and Cosine Series.*

1. Readers of the *Gazette* are indebted to Mr. Hardy for bringing this proof to notice. But I cannot follow Mr. Hardy in his praises of this method. It has a certain amount of gloss, but this is all on the surface. When you examine it closely you find that it evades the real difficulty which occurs in this line of proof.

2. I may explain my meaning by a reference to the Binomial Theorem. If m is rational, and $1>x>-1$, the Binomial Series for $(1+x)^m$ can be established purely from the Multiplication Theorem for two absolutely Convergent Series. A question of ambiguity arises in the course of the proof, but it does not require any knowledge of the Continuity of Series to determine it. (*Vide* my Note in the *Gazette*, January, 1906.)

If x is complex and $|x|<1$, m being rational as before, then a knowledge of the continuity of the power series is necessary to fix *which* of the values of $(1+x)^m$ is represented by the series in question.

Now we should consider objectionable any proof of the Binomial Theorem in these cases which burks the question of ambiguity. Exactly of this character is my main objection to Stolz and Gmeiner's proof. (In these remarks, it will be understood, I take Hardy for my text. I have not seen the original, nor do I know German.)

Further, Stolz and Gmeiner's proof seems to take the principle of the continuity of the power series as a matter of course. We can, indeed, obtain the theorem $\exp(ix)=\cos x+i\sin x$, without the use of this principle, if x is rational. But for a complete proof, the principle *is* required, and must not be assumed, without proof.

In the upshot, a complete and rigorous proof in this line is not so simple, nor the presuppositions involved so few, as might seem at first.

With these remarks, I shall attempt to supply what is lacking to make the proof satisfactory.

3. We require the following Lemma: If θ lies between 0 and $\frac{\pi}{2}$, and $\sin\theta<\frac{1}{m}$ where $m>1$, then $\theta<\frac{\pi}{2m}$.

This can be deduced from the theorem that $\frac{\sin x}{x}$ constantly decreases as x increases from 0 to $\frac{\pi}{2}$, as follows:

$$\frac{\pi}{2}>\frac{\pi}{2m};\quad\therefore\ \frac{\sin\frac{\pi}{2}}{\frac{\pi}{2}}<\frac{\sin\frac{\pi}{2m}}{\frac{\pi}{2m}};$$

$$\therefore\ \sin\frac{\pi}{2m}>\frac{1}{m};\ \text{ and }\ \therefore\ >\sin\theta.$$

Hence $$\frac{\pi}{2m}>\theta.$$

4. To prove the theorem, we first show that the modulus of $\exp(ix)$ is equal to unity for all real values of x; and therefore for every real value of x, $\exp ix=\cos\phi+i\sin\phi$ for a single value of ϕ in the interval 0 to 2π (0 included, 2π excluded).

Let $\exp i=\cos u+i\sin u$, where u has the unique value which is possible in the interval 0 to 2π.

Then $$\cos u=1-\frac{1}{\underline{|2}}+\frac{1}{\underline{|4}}-\dots,$$

and $$\sin u=\frac{1}{\underline{|1}}-\frac{1}{\underline{|3}}+\frac{1}{\underline{|5}}-\dots.$$

Since $\cos u$ and $\sin u$ are positive, we see the unique value of u above referred to lies in the narrower interval 0 to $\frac{\pi}{2}$.

Let now n be a positive integer >1, and let $\exp\frac{i}{n}=\cos\psi+i\sin\psi$, where ψ has the single value which is possible in the interval 0 to 2π. Then

$$\cos\psi=1-\frac{1}{\underline{|2}}\cdot\frac{1}{n^2}+\frac{1}{\underline{|4}}\cdot\frac{1}{n^4}-\ldots,$$

and

$$\sin\psi=\frac{1}{n}-\frac{1}{\underline{|3}}\cdot\frac{1}{n^3}+\frac{1}{\underline{|5}}\cdot\frac{1}{n^5}-\ldots.$$

Thus $\cos\psi$ and $\sin\psi$ are positive, and the latter is less than $\frac{1}{n}$. Hence ψ lies in the interval 0 to $\frac{\pi}{2}$ and its sine $<\frac{1}{n}$. Hence, by the lemma,

$$\psi<\frac{\pi}{2n}.$$

We thus have shown that

$$\exp\frac{i}{n}=\cos\psi+i\sin\psi,\quad\text{where}\quad 0<\psi<\frac{\pi}{2n}.$$

Raising each side to the n^{th} power,

$$\exp i=\cos n\psi+i\sin n\psi,$$

where, since ψ lies between 0 and $\frac{\pi}{2n}$, $n\psi$ lies between 0 and $\frac{\pi}{2}$.

Consequently $n\psi=u$, the unique value already referred to. Hence $\psi=\frac{u}{n}$, and therefore,

$$\exp\frac{i}{n}=\cos\frac{u}{n}+i\sin\frac{u}{n}.$$

We can now show that $u=1$. For, we have

$$\sin\frac{u}{n}=\frac{1}{n}-\frac{1}{\underline{|3}}\cdot\frac{1}{n^3}+\frac{1}{\underline{|5}}\cdot\frac{1}{n^5}-\ldots.$$

$$\therefore\ \frac{1}{n}>\sin\frac{u}{n}>\frac{1}{n}-\frac{1}{\underline{|3}}\cdot\frac{1}{n^3}.$$

$$\therefore\ 1>n\sin\frac{u}{n}>1-\frac{1}{\underline{|3}}\cdot\frac{1}{n^2}.$$

The middle term tends to the limit u, while the extreme terms both tend to the limit 1, when n is infinitely increased. Therefore $u=1$.

We thus get the result that if n be any positive integer,

$$\exp\frac{i}{n}=\cos\frac{1}{n}+i\sin\frac{1}{n}.$$

If now m be any positive integer, raising both sides to the power m, we get

$$\exp\left(i\frac{m}{n}\right)=\cos\frac{m}{n}+i\sin\frac{m}{n}.$$

This proves the theorem

$$\exp ix=\cos x+i\sin x,$$

when x is any rational positive quantity. Taking reciprocals, we see that the theorem is true when x is any rational negative quantity. The result

is thus true for any rational x. And the truth follows for irrational values of x by the continuity of $\exp ix$ on the one side (which has to be proved), and of $\cos x$ and $\sin x$ on the other.

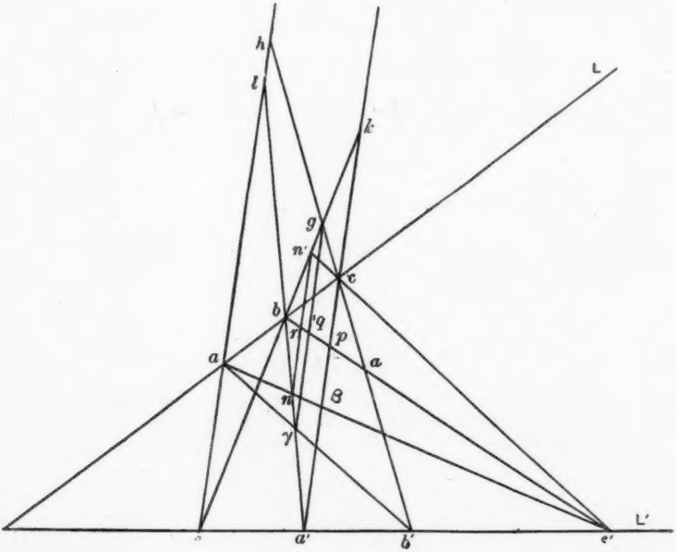
5. In conclusion, I wish to add a few words. The Multiplication Theorem of two absolutely convergent series is undoubtedly an important theorem, leading as it does to proofs of the Binomial and Exponential Theorems and of the sine and cosine series, not to mention its importance, otherwise, in theory. But there is a theorem of far greater importance in the Theory of Infinite Series and Infinite Products which appears neglected; and to this theorem, I think, we should turn for any substantial improvement in the elementary treatment of Infinite Series and Infinite Products. And I am of opinion it is high time we should give this theorem its fundamental place and relegate the Multiplication Theorem to second rank. I hope to return to this point as soon as possible. V. RAMASWAMI AIYAR.

Gooty, India,
24th January, 1907.

260. [L¹. 1. c.] *Pascal's Theorem.*

Proved for the conic and line-pair by the methods of Euclid and Apollonius.

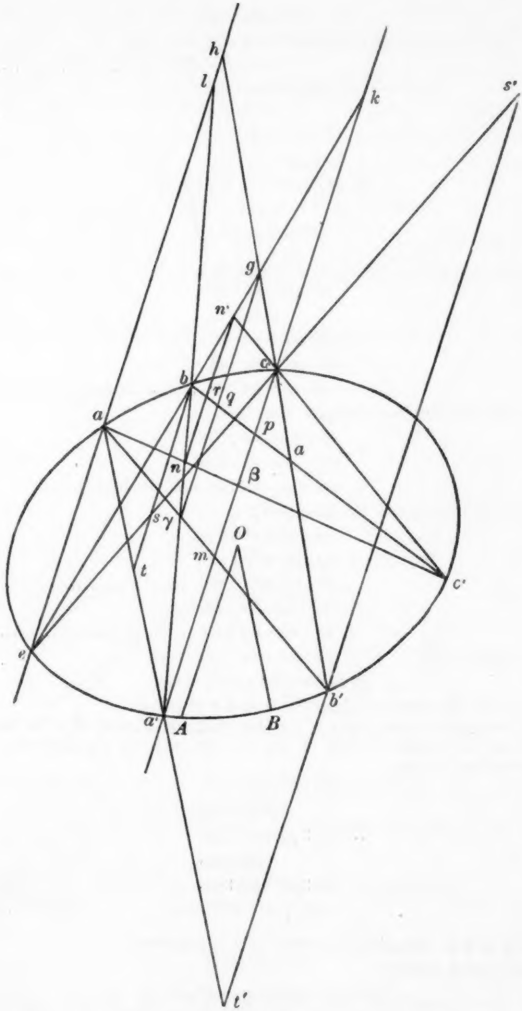
a, b, c and a', b', c' are any six points on a conic, or by threes on the pair of lines L and L'.



α, β, γ are the intersections of $(bc', b'c)$, $(ca', c'a)$, $(ab', a'b)$.

It is required to prove that α, β, γ are collinear.

Through a draw ae parallel to ca'. In the conic, O is the centre bst, $b's't'$, and OA are parallel to ca', and OB is parallel to aa'. Complete the figures by joining points as required.



I. For the conic. By similar triangles,

$$\begin{aligned} eh : b's' &= hc : cb' \\ &= am : mb' \\ &= aa' : a't \end{aligned}$$

and $$a'm : b't' = aa' : at'.$$

$$\therefore\ eh\,.\,a'm : aa'^2 = b's'\,.\,b't' : at'\,.\,a't'$$

$$= OA^2 : OB^2. \qquad \text{(1)}$$

Again, $$ck : sb = ec : es$$

$$= aa' : at.$$

$$\therefore\ ck : aa' = sb : at$$

and $$al : aa' = bt : ta'.$$

$$\therefore\ ck\,.\,al : aa'^2 = sb\,.\,bt : at\,.\,ta'$$

$$= OA^2 : OB^2. \qquad \text{(2)}$$

$\therefore$ by (1) and (2), $$ck\,.\,al = eh\,.\,a'm.$$

II. For the line-pair. By similar triangles,

$$ck : ae = kb : be$$

$$= ka' : le.$$

$$\therefore\ ck : ka' = ae : le.$$

$$\therefore\ ck : ca' = ae : al.$$

$$\therefore\ ck\,.\,al = ca'\,.\,ae. \qquad \text{(3)}$$

Again, from the similar triangles $b'eh$, $b'a'c$,

$$eh : a'c = b'e : b'a'$$

$$= ae : a'm.$$

$$\therefore\ eh\,.\,a'm = a'c\,.\,ae = ck\,.\,al, \text{ by (3).}$$

$\therefore$ in both the conic and the line-pair,

$$eh : ck = al : a'm.$$

$$\therefore\ eg : gk = l\gamma : \gamma a'.$$

$$\therefore\ ek : gk = la' : \gamma a' \qquad \text{(4)}$$

and $$bk : be = ba' : bl.$$

$$\therefore\ bk : ek = ba' : a'l. \qquad \text{(5)}$$

$\therefore$ by (4) and (5), $$bk : gk = ba' : \gamma a'.$$

$\therefore$ γg is parallel to $a'c$ or ae.

Now γ is the intersection of $(ab', a'b)$ and g of $(eb, b'c)$.

$\therefore$ if we introduce the point c' in the place of b', and if n be the intersection of $(ac', a'b)$, and n' that of $(eb, c'c)$, nn' will be parallel to $a'c$ or ae, and therefore also to γg.

$$\therefore\ \beta p : pc = nr : rn'$$

$$= \gamma q : qg.$$

$$\therefore\ \beta p : \gamma q = pc : qg$$

$$= \alpha p : \alpha q.$$

$$\therefore\ \beta p : p\alpha = \gamma q : q\alpha.$$

$\therefore$ α, β, γ are collinear.

JOHN J. MILNE.

261. [X. 4. b. β.] *Graphical solution of a biquadratic.*

The roots of the quartic

$$x^4 + ax^3 + bx^2 + cx + d^2 = 0$$

are the abscissae of the meets of the hyperbola $xy = d$ and the circle

$$x^2 + y^2 + ax + \frac{c}{d}y + b = 0.$$

The last term of the quartic can always be made positive by increasing the roots, each by the same amount, so that the method always applies. We get the same values of x by using the hyperbola $xy = -d$.

E. J. NANSON.

262. [C. 1. f.] *Note on Turning Values.*

The following three theorems are obvious to the graphist, all functions mentioned being continuous.

I. If $f'(a)=0$, $f''(a)\neq 0$, then $f(a)$ is a turning value.

II. If $f'(a)$ is a turning value, then $f(a)$ is an inflexion value.

III. If $f'(a)$ is a zero inflexion value, then $f(a)$ is a turning value.

Suppose now that $f_r(a)=0$ when $r=1, 2, 3, \ldots n-1$, but that $f_n(a)\neq 0$.

Then by I. $f_{n-2}(a)$ is a turning value;

$\therefore$ by II. $f_{n-3}(a)$ is a zero inflexion value;

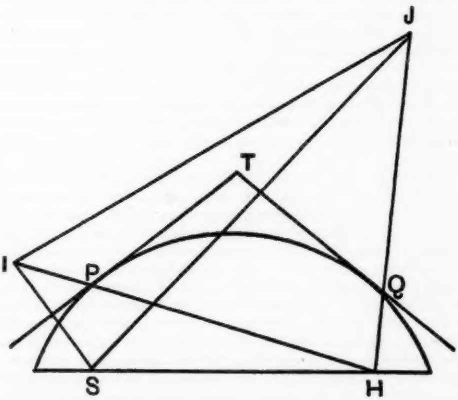
$\therefore$ by III. $f_{n-4}(a)$ is a turning value;

$\therefore$ by II. $f_{n-5}(a)$ is a zero inflexion value;

and so on, so that $f(a)$ is a turning value or an inflexion value according as n is even or odd.

E. J. Nanson.

263. [L[1]. 4. a.] *Alternative proof of a standard theorem in Geometrical Conics.*



Let S be one focus and I, J its respective images in the two tangents TP, TQ from an external point T to an ellipse.

Since $TI(=TS)=TJ$, and $HI(=2AC)=HJ$, therefore TH is perpendicular to (and bisects) IJ.

Moreover, TQ is perpendicular to SJ.

Therefore angle $HTQ=SJI=$half STI at the centre$=STP$.

W. P. Workman.

REVIEWS.

Cambridge Tracts in Mathematics and Mathematical Physics. (Cambridge University Press.)

No. 4. The Axioms of Projective Geometry. By A. N. Whitehead, Sc.D., F.R.S. 2s. 6d. net.

No. 5. The Axioms of Descriptive Geometry. By the same. 2s. 6d. net.

Among the interesting and important tracts which are being published by the Cambridge Press under the editorship of Messrs. J. G. Leathem and E. T.

Whittaker, these two are of special importance, since they give for the first time to English readers a general outline of abstract non-Euclidean geometry. It is a matter of satisfaction that the work has been entrusted to one who has a complete mastery of the logic of the subject. The reader of the pamphlets may find them difficult, and may well wonder whether many parts might not have been put in a simpler and more attractive form; but he can scarcely fail to extract from them many new and far-reaching ideas.

It may be conceded that an enquiry into the degree of simplicity and independence of any suggested system of geometrical axioms appears on the surface rather dry and repelling; but the pamphlets are not so much concerned with this aspect of the subject as with the consequences to which the axioms lead. Still, the reduction of axioms to their simplest terms, and their resolution into independent parts, is essential for the understanding of geometry, and for distinguishing between different geometries. A perfect system of axioms should be:—(1) complete, containing a formal statement of every assumption made; (2) consistent, containing no statements conflicting with one another or leading to contradiction; (3) independent, *i.e.* remaining consistent if any one axiom be replaced by its contradictory; and (4) elemental, *i.e.* completely resolved into separate elemental propositions. The object of discovering such a perfect system is to find out exactly how much is assumed in any particular geometry. When such a system has once been found it may be recast into an equivalent system, containing a much smaller number of axioms, and better adapted for geometrical reasoning. Such a revised system need only be complete and consistent. In the *Bulletin of the American Mathematical Society*, Vol. XIV. No. 6, a short abstract is given of a paper, not yet published, in which Profs. O. Veblen and J. W. Young claim to have given a completely independent system of axioms for the different types of projective geometry. The condition that a perfect system of axioms should be elemental is a rather vague one which can, perhaps, only be approximated to.

Whitehead, following Pieri for the most part, gives 19 axioms for projective* geometry, taking point and line as undefined. The first 10 axioms relate to the point and line, and lead to the conclusion that a line contains at least three points, and is determined by any two of its points. The 11th axiom states that the points of space do not all lie on one line. The 12th axiom says:—"If A, B, C are non-collinear points, and A' is a point on BC, distinct from B and C, and B' is a point on CA, distinct from C and A, then the lines AA' and BB' possess a point in common." Whatever may be thought of the first 11 axioms it seems obvious that this one is a compound axiom, and so does not belong to an elemental system. It contains in a single statement all the properties of the plane. The statement is not about one set of three non-collinear points, but about every such set. This axiom, or an equivalent one, appears in most systems that have been suggested, and seems to constitute a weak point in them.

The author then tells us (p. 8) what deductions may be made, such as the fact that two co-planar lines always intersect, etc. It is unfortunate at this early stage that the proofs are not given, as they could not have failed to enlighten the reader as to the meaning and bearing of the axioms, and the meaning of the terms used in them, which are not altogether free from vagueness.

These 12 axioms do not suffice to prove Desargues' theorem concerning perspective triangles; but it is shown that this theorem can be proved with the help of one more axiom, that all the points of space do not lie on one plane. Previously a descriptive definition of a harmonic range has been given, viz. that A, B, C, D are harmonic if A and C are the points of intersection of two pairs of opposite sides of a complete quadrangle, and B, D are the points in which the third pair of opposite sides intersect the line AC. In connection with this it is found that a new axiom (Fano's axiom) is required, viz. that if A, B, C are distinct, then D is distinct from B. In stating an equivalent form of this axiom the term quadrilateral is used where quadrangle is apparently meant (p. 15). It is proved by Desargues' theorem that if A, B, C are given then D is fixed, and

* From here onwards we use the terms projective and descriptive in the sense of the author. The distinction made between them is that in projective geometry and projective space two co-planar lines have always one, and only one, intersection; in descriptive geometry and descriptive space they have one intersection or none.

also that B, C, D, A are harmonic; but it does not seem to be shown that if one pair of corresponding points, A, C or B, D, coincide, one point of the other pair coincides with them, a fact which appears to be assumed later (p. 24).

These axioms are still insufficient to prove the fundamental theorem, viz. that if the range of points on one line corresponds projectively (that is, through any number of projections) to the range of points on another, and if three pairs of corresponding points are known, then the correspondent of any fourth point is uniquely determinable. It is shown that the fundamental theorem will follow if Pappus' theorem (Pascal's theorem for the case of two lines) is assumed. But this would be introducing another distinctively compound axiom; and fortunately it can be avoided. This is done by introducing three axioms of order and a final axiom expressing the Dedekind property. This question of order as represented here leaves nothing to be desired in the matter of dryness. The following chapter on quadrangular involutions would have gained in interest if it had been confined to the important properties of prospectivities, a name given to any projective transformation of a line into itself so that there is one, and only one, self-corresponding point. By means of these properties it is shown how numerical coordinates can be assigned to the points of space in such a way that planes and lines have linear equations; and that for any such system of coordinates a certain function of the coordinates of 4 given points on a line, called the anharmonic ratio of the 4 points, remains invariant for the points, and for any projective transformation of them. The establishment of these results forms the chief aim and climax of the tract on Projective Geometry. On p. 50 a very complicated construction is given for a point which is merely a harmonic conjugate.

In the tract on Descriptive Geometry a different set of axioms is chosen as the foundation, preference being given to the system due to Veblen. It is shown in Chapter II. that these axioms hold for any convex region of projective space;* and it is thus demonstrated that the descriptive geometry (properly so called) of any convex portion of a projective space is descriptive geometry in the sense of the author. This relation between descriptive and projective geometry is nevertheless *practically* inoperative, since it does not hold in metrical geometry. The author then considers the converse question of building up an ideal projective space of which any given descriptive space shall be a part; and in doing so introduces us to a truly bewildering notation, with the object, apparently, of satisfying the demands of formal logic, in which he does not appear to be successful. The tract goes on to consider the collineation or general projective transformation, the congruence transformation, the theory of the absolute, and metrical geometry. In regard to the collineation the author says (p. 43) that a certain system of equations is consistent under certain conditions; this is not so if the origin transforms to a point in the infinity plane. This illustrates the fact that absolute reliance cannot be placed on many of the statements made. The usual method is followed of treating Euclidean space as a limit of projective or descriptive space, in order to obtain its metrical properties. Such a method is legitimate for research, but quite invalid for proof.

The conspicuous merits of the two tracts are marred by some serious defects. Nothing is done to make the subject as simple as possible; no emphasis is laid on the important facts; no italics and no headings, except to chapters, are used; there is not even a sign in the first of the two tracts to show where a definition is given; there is practically no attempt to make the thread of the argument clear; there is no summing up and no index. The consequence is that the tracts fail in great measure of their object, which is to serve as an introduction; they are too involved and difficult.

The author shows a strange disinclination to make use of recognised nomenclature; and although he may not be the first to make undesirable changes which can only retard progress, he must share the responsibility. The terms elliptic and hyperbolic, applied to geometry or space, may not be altogether free from objection; but they are preferable to projective and descriptive, by which a distinction is inferred which has no existence. One can only protest when a most useful and suggestive term like projective is altered or restricted to a meaning

* Not apparently without exception, however; for Axiom V. (p. 8) says, "If A, B are distinct points there exists a point C such that A, B, C are in the order ABC." This is not true if B is on the boundary of the convex region.

of which the word itself affords no indication. We have already referred to the bewildering nomenclature for ideal elements. We believe that this is due to an essentially wrong conception of the subject, though that may be a matter of opinion. From our point of view all desiderata are met by the terms actual and ideal. The author, however, requires the terms projective, proper, improper, coherent, and associated; and gives us definitions which are almost incomprehensible. Several other instances might be given of minor importance.

Some passages are obscure, as in the following important statement taken from the first chapter:—"Geometry, in the widest sense in which it is used by modern mathematicians, is a department of what in a certain sense may be called the general science of classification. This general science may be defined thus: given any class of entities K, the sub-classes of K form a new class of classes, the science of classification is the study of sets of classes selected from this new class so as to possess certain assigned properties. For example, in the traditional Aristotelian branch of classification by species and genera, the selected set from the class of sub-classes of K are (1) to be mutually exclusive, and (2) to exhaust K; the sub-classes of this set are the genera of K; then each genus is to be classified according to the above rule, the genera of the various genera of K being called the various species of K; and so on for sub-species, etc." This may be clear to a logician, but we confess we don't understand it, and that it gives us the feeling of having things thrown at us. It seems a very incomplete description of Geometry to say that it is merely concerned with classification. Geometry is concerned with operations which lead to classification, but the operations are distinct from the classification which results from them. This idea crops up again in the statement (p. 44), "Prospectivities on the same line with the same self-corresponding point have the properties of a system of magnitudes." This can scarcely be true of the prospectivities themselves, which are mere transformations or correspondences, but it may be true of the operations which produce the transformations.

In agreement with many others the author states that lines and planes are classes of points. It is easy to show that this point of view is inconvenient and unnecessary. It is inconvenient because it destroys the principle of duality. It is unnecessary because it is just as easy to say that a line is an entity associated in a one-one relation with a class of points (or a class of planes); and similarly for a plane, and also for a point. It seems as if the author and others sometimes regard such a relation as involving identity, and at other times not. A point is associated in a one-one relation with a class of lines, and also with a class of planes; but the point and the class of lines and the class of planes are not considered identical. If this were so a point would be the class of all points in space. In the discussion of ideal points the author appears to fall into this very error. He defines a projective point and an ideal point as the same thing, viz. a certain class of actual lines; and a projective line and a projective plane as classes of projective points. According to these definitions a projective plane is the class of all lines in space, and is indistinguishable from any other projective plane. This conclusion seems unavoidable unless by a projective point is understood a single entity standing in a one-one relation to a class of lines; but this is absolutely precluded by the definition of a projective point. These entities, which we call ideal points, have to be invented sooner or later; and it is better to invent them from the very beginning, when all the author's complex nomenclature becomes unnecessary.

From the abstract point of view it is natural that there should be no reference to Riemannian space; but from the metrical and objective point of view its omission is to be regretted. Riemannian space only differs from the simple projective space of the author in that two co-planar lines have two (antipodal) intersections instead of one. Analytically there is practically no difference, as a pair of antipodal points have the same coordinates. If space is objective it is much more natural to say that Euclidean space is the half of a Riemannian space, than to say it is a projective space deprived of a plane. Points at infinity in opposite directions in Euclidean space would then not be the same ideal point, but a pair of antipodal ideal points. Riemannian space and descriptive space can both be objectively modelled in objective Euclidean space, but simple projective space cannot be; and there is no proof that simple projective space is objectively possible.

F. S. M.

Introduction to Higher Algebra. By MAXIME BÔCHER. Pp. xi, 321. New York: The Macmillan Company. 1907.

Much east wind has blown across the North Sea on our insular mathematics since Salmon wrote his *Lessons Introductory to the Modern Higher Algebra*, and it is quite time that English-speaking students should be provided with a book, bearing much the same title, and containing many of the same truths, but presenting these truths less as they sprang to light than as the prevailing current exposes them. Without it they are at a great disadvantage in their efforts to learn from the teachers of the day, influenced as these are mainly by those teachers of former days who till the wind blew were too little heeded here, and to express themselves effectively. The book is now before us, and it is a good book. In reviewing it, I am conscious of a certain conservative prejudice, but am anxious not to be thought a hostile critic. Only one paragraph below is intended to be read as a complaint. Dr. Bôcher has succeeded in making an unfamiliar aspect, of things mostly familiar, quite pleasing. His explanations and proofs are at once brief and clear. In revision for the press he and his collaborator, Mr. Duval, must have been most patient. His printers have produced a piece of typography than which nothing better could be desired. Every superscript and suffix is not only printed right but printed legibly, so that an old-fashioned person who likes when possible to say β rather than $a_2^{(2)}$ finds himself almost converted to modern usage as he reads. Lastly, his publishers have done wisely in issuing the book at a reasonable price. It will be largely bought and widely read, as it deserves, instead of being only placed in libraries for occasional reference.

As little as ten years ago the feat of producing a book in English, introductory to the advanced study of algebraic polynomials, in which from end to end all allusion to Salmon should be avoided, would have been regarded as presumptuous to think of and impossible to accomplish. But "the old order changeth, yielding place to new," and we have such a book. Put aside then all notion of finding traditionally British ways of thinking set before you as models. Other ways are for the moment at any rate more scientific ways. It may be that even they are not perfect or immeasurably superior ways: it may be that a conflict of currents may yet end in a calm in which our home growths may stand erect and invigorated: but meanwhile as these bend before the blast Professor Bôcher has much to show us near their roots.

We used in what we called Higher Algebra to allow free use of the notation and some little of the procedure of infinitesimal analysis, but to disguise, if possible even from ourselves, that help from geometry was present to us—was this last scrupulosity pedantic? Now—and is this in its turn pedantry?—we must never allow that we are guided by infinitesimal analysis, but on the other hand may use geometric language with freedom, even to the nth dimension. Perhaps in some happy day to come the language of geometric neighbourhoods and that of differential coefficients may both be allowed to do good service in Algebra, by leading along paths of least resistance to desired goals.

The one complaint which I have to make as to a regrettable omission from the book appears to be connected with the author's fidelity to a narrowness of view as to what is lawful in Algebra. Once we used, with loose brevity, to speak of every equation having a root. Nowadays, to be neither inexact nor verbose, we do not describe what we mean, but talk of *the fundamental theorem of Algebra*. Dr. Bôcher does, like everyone else. His first chapter is on polynomials and their most fundamental properties. He states and names the theorem, says he shall use it (though not much!), and does not prove it. This is a great pity. If the theorem is really fundamental, how can the absence of a proof of it be reconciled with the statement in his preface that he intends "to introduce the student to Higher Algebra in such a way that he shall learn what is meant by a proof in Algebra, and acquaint himself with the proofs of the most fundamental facts," or again with the claim "throughout to lay a sufficiently broad foundation to enable the reader to pursue his further studies intelligently"? It cannot be that the proof fails to appear for the school-book reason that it is difficult, and might block the way of junior students in their first advances. No! the reader soon finds out that it is not the way of the book to shirk difficulties. The reasons for the omission are

given on p. 17, and are nothing if not paradoxical. "This theorem," we read, "fundamental though it is, is not necessary for most of the developments of this book. Moreover the methods of proving the theorem are essentially not algebraic, or only in part algebraic." This is strangely like saying that to correct the name *the fundamental theorem of Algebra* we ought to omit the words *the* and *theorem of Algebra*, leaving the remaining word *fundamental* secure but liable to misinterpretation. Few who read the book can admit that the theorem is not essentially a part of the "broad foundation" of the subject which the preface is naturally read as meaning to lay; and, as to Dr. Bôcher's second dictum, can he seriously mean that no algebraical proofs of the theorem exist, even giving a narrow meaning to the word *algebraical*? If so, the sooner a new definition of the word is agreed upon, which will admit into Algebra its own fundamental theorem, the better.

The complaint over, we can pass on rapidly. After the first chapter a few follow which can be grouped as determinantal. Of these III. and IV., on linear dependence and systems of linear equations, will be especially useful to English readers. Our books have not often contained a sufficiently complete and well-arranged statement of the beautiful simplicity of the facts as to m equations in n unknowns; but the slighting reference to these books, in respect of the chief case when $m=n$ and the equations are independent, which occurs on p. 43, is superfluous.

The introduction of linear transformations in Chap. VI. proceeds in the manner, not very natural to most of us, which has the support of Lie, and other very high authorities, who seem to have thought first of displacements of marking points rather than of changes of reference. In the following quotation the italics are mine. "We have occasion to introduce, in place of the unknowns, or variables, we had originally, *certain functions of these quantities* which we regard as *new* unknowns or variables." Sylvester used to call this process *substituting* as distinguished from *transforming*. Our notion that the easier first idea in transformation is that of replacing the *old* by expressions in terms of the *new* is inveterate. It is interesting to notice that Professor Bôcher sometimes comes over to our side when he has actually to apply linear transformations. See for instance p. 127.

In Chap. VII. invariants are first introduced. All intention of giving a comprehensive treatment of them is disclaimed, and rightly, seeing that the book is to be an introduction to Higher Algebra, and not a compendium of any branch of it. Those to whom an invariant has hitherto always meant a function of the coefficients of a form, which has the invariant property for linear transformations of the variables in that form, will find much to profit from in the exposition given of the idea of invariancy in general.

Bilinear and quadratic forms are dealt with at some length in the valuable chapters VIII. to XIII.

Chaps. XIV. to XVI. deal with the reducibility of polynomials, and with questions of common factors. Resultants are led to from the algorithm of the Greatest Common Divisor. Their interpretation is not perhaps exhibited with so complete a freedom from obscurity as are most things in the book, and the determinant on p. 195 does not convey its meaning at a glance. However, the determinant appears again, and this time clearly, a few pages later in connection with a description of Sylvester's dialytic method, which is apparently introduced mainly in order to call attention to the defectiveness of the proof usually associated with the method. It is a pity that the few lines necessary to remedy this defectiveness have not been supplied. The facts given with regard to sub-resultants in § 69 are interesting.

Afterwards come another chapter on invariants, and a pleasing one, which might have been a little more developed, on symmetric polynomials.

So far in reading the later chapters, on polynomials, one has had a feeling that powerful machinery, of matrices, etc., has been constructed earlier in the work, and is lying idle while quite simple matters are being considered, professedly without exhaustiveness. In the final three chapters, on λ-matrixes and the classification of bilinear and quadratic forms, we see the machinery at work. They are good and important chapters; and the first of them especially combines generality with as easy intelligibility as the subject admits of. It is no disparagement to the others to say that the presentation of their

matter in Bromwich's recent tract will be more easily read by those who like to be helped on from the simple to the general rather than to master the general and exemplify it by the particular. Bôcher's dissatisfaction with Bromwich for his use of the term *invariant factor* instead of *elementary divisor* appears reasonable. E. B. ELLIOTT.

Beiträge zur Theorie der Linearen Transformationen. Von W. SCHREIBNER. 1907. (Teubner, Leipzig.)

This book consists of four chapters, occupying 150 pages, followed by an Appendix of four sections, occupying 90 pages altogether, and a full analytical table of contents, which last greatly helps the perusal. It is a collection of tracts, or didactic monographs on parts of a subject, rather than one sustained treatise.

The first chapter provides those principles of the algebraical theory of invariants and covariants of binary forms to which attention is going to be confined. Forms are taken as non-homogeneous in one variable, and not as homogeneous in two. A more remarkable departure from usage, in Germany at any rate, is that the treatment of the subject is non-symbolical. Moreover no prominence is given to the use of differential operators. It follows that the theory of complete irreducible systems of concomitants cannot enter. That such systems are finite is a fact just mentioned as one of which the proof is beyond the scope of the work. The chapter culminates in a tract on the expression of covariants in terms of *algebraically* complete systems, or more precisely, to use Sylvester's word, in terms of systems of *protomorphs*. To our author the one authority on this subject is Hermite.

The second chapter contains a somewhat full treatment of the theory of quadratic, cubic and biquadratic equations.

The third, a shorter one, applies the theory of the cubic and biquadratic to the reduction of elliptic differential elements.

The fourth, and last, chapter deals with the quintic and sextic—not, of course, with the syzygetic theory of their concomitants, but with the bearing of the protomorphic theory of Chap. I. upon them—and with the subject of sextic (here by preference called bi-cubic) resolvents. Expressions for important invariants and covariants are given both in terms of the coefficients and in terms of the roots, occasionally with the singularly candid reservation that those in terms of the roots need further verification: but this reservation of course only accompanies certain expressions on the accuracy of which no argument is made to depend. An interesting part of the chapter is that in which Hermite's canonical forms of the quintic and sextic, in which the second and fourth coefficients are zero and the remaining coefficients invariants, are obtained by means of a theorem, quoted as given without demonstration by Brioschi, and now proved, that, if h and g are two covariants of one order of a binary m-ic in x, the transformation $x'=\frac{h}{g}$ produces from that m-ic one in x' with invariant coefficients.

The first section of the Appendix is a short geometrical tract dealing with the transformation of circles into circles by means of the bilinear relation $apq+bp+cq+d=0$ between complexes p, q. The second is a full carrying out of a Tschirnhausen transformation used by Hermite. The third and fourth, on the Icosahedral equation, and the linear transformation of Theta and Elliptic Modular functions, respectively, contain much instructive analysis and a considerable array of results; but the latter at any rate makes so considerable an appeal to analytical conceptions not provided in this book that it would seem rather to belong to another. Much more space, elsewhere in the book as well as here, might with advantage have been devoted to clear explanation of what is sought for by analytical manipulation, and of the interpretation of conclusions.

The book probably lays no claim to be one of essential origination, but it is unconventional, and is certainly one of patient independent investigation. As a whole it lacks the form and finish which might have made it attractive, but it consists of highly instructive parts which will repay study. It is fragmentary, but very far from trifling. E. B. E.

An Introduction to the Theory of Multiple Periodic Functions. By H. F. BAKER. Pp. vii+335. (Cambridge University Press.)

After the great advance made in the theory of doubly periodic functions when Jacobi introduced his theta functions and gave their expansions, the q-series, it occurred to Göpel and Rosenhain that functions of more variables expressed by series formed on the same lines might be made to play an important part in the theory of hyperelliptic functions. The writers named carried out their idea in the case of two variables, but the extension to a greater number was only successfully made by Riemann and Weierstrass.

The following sketch of the outstanding points in the theory may be of interest to readers of the *Gazette*:

(1) In a rational algebraic function of two or more variables two kinds of singularity occur, namely, in the first place, the pole or infinity where the denominator vanishes and the numerator does not; and secondly, the point of indetermination where numerator and denominator both vanish, but have no common factor, as for instance, the origin in the case of the functions x/y, $(x^2+y^2)/(x-y)$.

(2) If a function of p variables, $x_1, x_2 \ldots x$, has, for finite values of these, only such singularities as occur in rational algebraic functions, it cannot have more than $2p$ periods.

A period of the function $f(x_1, x_2 \ldots x_p)$ means a set of constant quantities a_1, $a_2 \ldots a_p$, such that

$$f(x_1+a_1,\ x_2+a_2 \ldots,\ x_p+a\ )=f(x_1,\ x_2 \ldots x_p)$$

for all values of the variables. The function $\tan x$, for instance, has the period π since $\tan(x+\pi)=\tan x$: the elliptic functions are functions of one variable having two periods.

(3) Just as by integrating the irrational algebraic function $(1-x^2)^{-\frac{1}{2}}$ we are led to the inverse of a periodic function, the sine, so by considering the integrals of more complicated irrational expressions we are led to the inverses of functions of p variables with $2p$ periods, known as Abelian functions: hyperelliptic functions are a special case.

(4) Functions of p variables with $2p$ periods can also be constructed as quotients of the so-called multiple Θ functions, referred to above.

(5) In both cases, (3) and (4), the periods are restricted by conditions. For instance, it is not possible by either method to construct a function of two variables x_1, x_2, having four periods (a_1, a_2), (b_1, b_2), (c_1, c_2), (d_1, d_2), if these eight quantities $a_1, a_2 \ldots$ are arbitrarily chosen. The restrictions under (3) are as a rule more severe than those under (4), but when $p=2$ the two methods have the same scope.

Functions with periods of the most general kind can be constructed, but their singularities are not all of the algebraic type: for instance, in the case of two variables, such a function is $\Sigma \operatorname{cosec}(x_1+m\lambda_1+n\mu_1)\operatorname{cosec}(x_2+m\lambda_2+n\mu_2)$, in which λ_1, λ_2, μ_1, μ_2 are complex quantities and the summation is over all integral values of m, n.

(6) Two questions arise: (a) Are the restrictions on the periods in (4) necessary if the singularities of the function are to be algebraic? (b) Could all the functions constructed under (4) be included by any means under (3)? To each the answer is yes, and the discussion takes up the second part of the work before us.

The first part contains the theory of functions of two variables with four periods, in a form simple but up to date, and especially in its connexion with two quartic surfaces, those of Kummer and Weddle, which have to these functions somewhat the same relation as the plane cubic has to elliptic functions. The system of conicoids through six given points is very prominent in the treatment. Very much space is saved throughout the work by the matrix notation.

There can be no doubt that the publication of such a book as this will, as the author laudably hopes, "encourage a wider cultivation of the subject" and "a wider use of the functions in other branches of mathematics."

A. C. DIXON.

QUERIES.

(50) [p. 231. Note 256.] This has been attributed to Routh or Webb. Is there any ground for this? $D_1 \ldots D_4$.

(51) Using only the ordinary school text-books on Mechanics, give an *à priori* reason in the case of the impact of two elastic spheres for *not* using $Ps=\frac{1}{2}mv^2-\frac{1}{2}mu^2$ and Newton's Law and then deducing a change of momentum, rather than using $Pt=mv-mu$ and deducing a loss of energy. A. C. J.

(52) Wanted a general solution of $x^2+y=a$; $y^2+x=b$. Titus.

(53) Find geometrically the ex- and in- centres of similitude of the circum- and 9 pt.- circles of a triangle. G.H.

(54) What curves are of the same degree as their pedals? B. E. N.

(55) What English medical men have been mathematicians of any note? M.D.

(56) Where can I find a discussion of triangles inscribed in an ellipse and (1) of maximum area; (2) of maximum perimeter? M. T. E.

(57) Wanted, a list of recent papers dealing with the mathematics of aviation. A. Bird.

ANSWERS TO QUERIES.

[38, p. 212.] Let AB be the given base bisected at C, DBC the given difference between the angles at the base, m the given median.

Find a 3rd proportional n to m and CB so that

$$mn=CB^2,$$

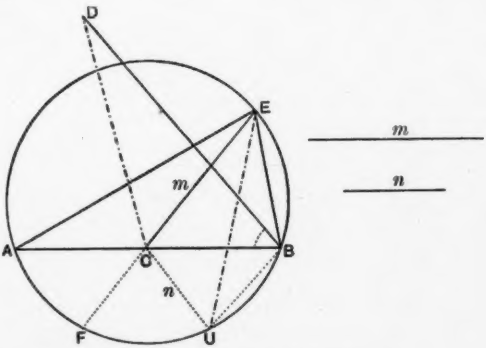
and in BD find a point D such that

$$n : m = CB : CD.$$

Bisect the angle BCD and make the bisecting line CE= the given line m.

AEB is the $\triangle$ required.

Proof. Describe the $\odot$ AEB, produce EC to meet the circumference in F, then $CF=n$.



Make the $\angle BCU=ACF$, U being on the circumference.
Then as C is the mid-point of the chord,

$$CU=CF=n.$$

The $\angle$s DCB, ECU being each double of ECD are equal; also the sides about them are proportional;

$$\therefore \text{ the } \triangle\text{s are similar and } \angle DBC = EUC.$$

But the $\triangle$s ECB, BCU are also similar, and $\therefore \angle EBC = BUC$.

Therefore also the remainders are equal,

$$i.e. \ EBD = EUB = EAB,$$

$\therefore$ the difference between $\angle$s EBA, EAB is DBA the given difference.

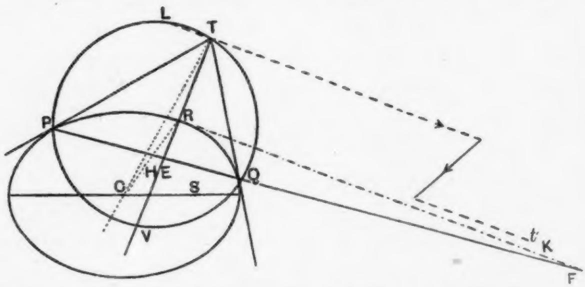
E. P. Rouse.

[39, p. 212.] Let TP, TQ be the given tangents and let the given normal meet PQ in E.

Describe the $\odot$ TPQ, draw the median TC and find U the end of the symmedian chord.

Then if C the centre can be found the solution is easy; for it is proved in No. 4 of the *Gazette* that the major axis of the conic bisects the $\angle TCU$, and that S being a focus $CS^2 = CU \cdot CT$.

F the pole of the line TE can be found by harmonic section. Draw $FR \perp TE$, then R is the point of the conic at which TE is a normal.



Let the $\perp$ at T to TE meet the chord PQ in K, and find H the pole of TK; then the chord of the conic of which H is the middle point will be $\parallel TK$ and $\therefore \parallel$ the tangent at R.

Join RH, and produce to meet the median TC at C. C is the required centre.

[The readiest way of finding the pole of *any straight* line through T (say TL) is to join U to the point L where TL cuts the circumcircle. UL meets the chord PQ at the required point. A geometrical proof of this statement is not difficult.]

E. P. Rouse.

BOOKS, ETC., RECEIVED.

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